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Bianchi Type-III Cosmological Models with Time Dependent Displacement Vector for Barotropic Fluid Distribution in Lyra Geometry

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Abstract Bianchi Type-III cosmological models for perfect fluid distribution with time dependent displacement field in the framework of Lyra geometry are investigated. To get the deterministic model of the universe, we have assumed two conditions (i) shear (σ) is proportional to the expansion (θ). This leads to $B = C^n$ where B and C are metric potentials and n is a constant. (ii) Universe is filled with barotropic fluid distribution which leads to $p = \gamma\rho$, $0 \le \gamma \le 1$, p being isotropic pressure and ρ the energy density. The physical and geometrical aspects of the model with a special case and singularities in the models are also discussed.

Keywords Bianchi III · Displacement vector · Lyra geometry

1 Introduction

Einstein introduced his General Theory of Relativity in which gravitation is described in terms of geometry and it motivated him to geometrize the other physical fields. Weyl [1] made one of the best attempts in this direction. He introduced a generalization of Riemannian geometry in an attempt to unify gravitation and electromagnetism. Weyl's theory was not taken seriously because it was based on the non-integrability of length transfer. Later Lyra [2] suggested a modification of Riemannian geometry which has a close resemblance to Weyl's geometry. In Lyra's geometry, the connection is metric preserving as in Riemannian geometry and length transfer are integrable. Lyra introduced a gauge function which removed the non-integrability condition of the length of a vector under parallel transport. Thus Riemannian geometry was modified by Lyra and was given a new name, called Lyra's geometry.

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Sen [3], Sen and Dunn [4] introduced a new scalar tensor theory of gravitation and obtained the field equations analogous to the Einstein's field equations based on Lyra's geometry which in normal gauge can be written in the form

$$R_{ij} - \frac{1}{2}Rg_{ij} + \frac{3}{2}\phi_i\phi_j - \frac{3}{4}\phi_k\phi^k g_{ij} = -\frac{8\pi G}{c^4}T_{ij}$$

where ϕ_i is the displacement vector and other symbols have their usual meaning. Halford [5] in his investigations, has pointed out that the constant displacement vector field ϕ_i in Lyra's geometry, has the role of cosmological constant Λ in General Relativistic treatment. Soleng [6] has investigated that constant gauge function ϕ in Lyra's geometry either included a creation field and is equal to Hoyle creation field cosmology [7, 8] or contains a special vacuum field which with the gauge vector term, may be considered as a cosmological term. Beesham [9] investigated FLRW cosmological models in Lyra's manifold with time dependent displacement field. The models so obtained, solve the singularity, entropy and horizon problems which exists in standard models of cosmology based on Riemannian geometry. Singh and Singh [10–12] have investigated Bianchi Type I, III and Kantowski-Sachs cosmological models with time dependent displacement field and have made a comparative study of R-W models with constant deceleration parameter in Lyra's geometry. Singh and Singh [13] have also investigated Bianchi Types I, V and VI_0 cosmological models in Lyra geometry. Bhowmik and Rajput [14] have investigated anisotropic Bianchi Type I cosmological models based on Lyra geometry considering deceleration parameter constant and time dependent. Pradhan et al. [15–19] and Rahaman et al. [20–22] in a series of papers have investigated cosmological models based on Lyra's geometry with constant and time dependent displacement field in different contexts. Mohanty et al. [23] have investigated a five dimensional perfect fluid cosmological model within the framework of Lyra Geometry. They have pointed out that neither perfect fluid nor dust distribution survive. They have also obtained exact solution of the vacuum field equations in context of Lyra Manifold. Recently Bali and Chandnani [24] have investigated Bianchi Type I cosmological model with time dependent gauge function for perfect fluid distribution within the framework of Lyra Geometry. To get the deterministic model, they have also assumed that eigen value σ_1^1 of shear tensor σ_i^j is proportional to the expansion θ.

2 The Metric and Field Equations

We consider Bianchi Type-III metric in the form

$$ds^2 = -dt^2 + A^2dx^2 + B^2e^{-2\alpha x}dy^2 + C^2dz^2 \tag{1}$$

where A, B, C are functions of t-alone.

Energy momentum tensor T_i^j for perfect fluid distribution is given by

$$T_i^j = (\rho + p)v_i v^j + pg_i^j \tag{2}$$

where $v_i = (0, 0, 0, -1)$; $v^i v_i = -1$; $\phi_i = (0, 0, 0, \beta(t))$; $v_4 = -1$; $v^4 = 1$; p is the isotropic pressure, ρ the energy density, v^i the fluid flow vector and β the gauge function.

Einstein's modified field equation in normal gauge for Lyra's manifold obtained by Sen [3]

$$R_i^j - \frac{1}{2}Rg_i^j + \frac{3}{2}\phi_i\phi^j - \frac{3}{4}\phi_k\phi^k g_i^j = -KT_i^j \tag{3}$$

(in geometrized unit where $G=1, c=1$) leads to

$$\frac{B_{44}}{B}+\frac{C_{44}}{C}+\frac{B_4C_4}{BC}+\frac{3}{4}\beta^2=-Kp, \tag{4}$$

$$\frac{A_{44}}{A}+\frac{C_{44}}{C}+\frac{A_4C_4}{AC}+\frac{3}{4}\beta^2=-Kp, \tag{5}$$

$$\frac{A_{44}}{A}+\frac{B_{44}}{B}+\frac{A_4B_4}{AB}-\frac{\alpha^2}{A^2}+\frac{3}{4}\beta^2=-Kp, \tag{6}$$

$$\frac{A_4B_4}{AB}+\frac{B_4C_4}{BC}+\frac{A_4C_4}{AC}-\frac{\alpha^2}{A^2}-\frac{3}{4}\beta^2=K\rho, \tag{7}$$

$$\alpha\left(\frac{A_4}{A}-\frac{B_4}{B}\right)=0 \tag{8}$$

where we have assumed $K=8\pi$.

The energy conservation equation $T^j_{i;j}=0$ leads to

$$\rho_4+(\rho+p)\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)=0 \tag{9}$$

and conservation of L.H.S. of (3) leads to

$$\left(R^j_i-\frac{1}{2}Rg^j_i\right)_{;j}+\frac{3}{2}(\phi_i\phi^j)_{;j}-\frac{3}{4}(\phi_k\phi^k g^j_i)_{;j}=0 \tag{10}$$

which leads to

$$\frac{3}{2}\phi_i\left[\frac{\partial\phi^j}{\partial x^j}+\phi^l\Gamma^j_{lj}\right]+\frac{3}{2}\phi^j\left[\frac{\partial\phi_i}{\partial x^j}-\phi_l\Gamma^l_{ij}\right]-\frac{3}{4}g^j_i\phi_k\left[\frac{\partial\phi^k}{\partial x^j}+\phi^l\Gamma^k_{lj}\right]$$
$$-\frac{3}{4}g^j_i\phi^k\left[\frac{\partial\phi_k}{\partial x^j}-\phi_l\Gamma^l_{kj}\right]=0 \tag{11}$$

Equation (11) is automatically satisfied for $i=1,2,3$.

For $i=4$, (11) leads to

$$\frac{3}{2}\beta\left[\frac{\partial}{\partial x^4}(g^{44}\phi_4)+\phi^4\Gamma^4_{44}\right]+\frac{3}{2}g^{44}\phi_4\left[\frac{\partial\phi_4}{\partial t}-\phi_4\Gamma^4_{44}\right]-\frac{3}{4}g^4_4\phi_4\left[\frac{\partial\phi_4}{\partial x^4}+\phi^4\Gamma^4_{44}\right]$$
$$-\frac{3}{4}g^4_4g^{44}\phi_4\left[\frac{\partial\phi_4}{\partial t}-\phi^4\Gamma^4_{44}\right]=0 \tag{12}$$

which leads to

$$\frac{3}{2}\beta\beta_4+\frac{3}{2}\beta^2\left(\frac{A_4}{A}+\frac{B_4}{B}+\frac{C_4}{C}\right)=0 \tag{13}$$

3 Solution of Field Equations

For the complete determination of the model, we assume that shear (σ) is proportional to the expansion (θ).

This leads to

$$B = C^n \tag{14}$$

The motive behind assuming this condition is explained as: Referring to Thorne [25] the observations of the velocity-redshift relation for extragalactic sources suggest that the Hubble expansion of the universe is isotropic today to within 30 percent [26, 27]. More precisely, the red-shift studies place the limit $\frac{\sigma}{H} \leq 0.30$, where σ is shear and H is Hubble constant. Collins et al. [28] have pointed out that for spatially homogeneous metric, the normal congruence to the homogeneous hyper-surface satisfies the condition $\frac{\sigma}{\theta}$ is constant where σ is shear and θ is expansion in the model.

Now (8) leads to

$$\frac{A_4}{A} = \frac{B_4}{B} \tag{15}$$

which leads to

$$A = lB \tag{16}$$

l being constant of integration.

Now (4) and (5) lead to

$$\frac{B_{44}}{B} - \frac{A_{44}}{A} = \frac{C_4}{C}\left(\frac{A_4}{A} - \frac{B_4}{B}\right) \tag{17}$$

Equation (14) leads to

$$\frac{B_{44}}{B} = n\left(\frac{C_{44}}{C} - \frac{C_4^2}{C^2}\right) + n^2\frac{C_4^2}{C^2} \tag{18}$$

From (5) and (6) we have

$$\frac{C_{44}}{C} - \frac{B_{44}}{B} + \frac{A_4 C_4}{AC} - \frac{A_4 B_4}{AB} + \frac{\alpha^2}{A^2} = 0 \tag{19}$$

Using (14), (15) and (18) in (19) we have

$$\frac{C_{44}}{C} + 2n\left(\frac{C_4}{C}\right)^2 = -\frac{\alpha^2}{l^2(1-n)C^{2n}} \tag{20}$$

Equation (20) leads to

$$\frac{d}{dC}(f^2) + 4n\frac{f^2}{C} = -\frac{2\alpha^2}{l^2(1-n)C^{2n-1}} \tag{21}$$

where $C_4 = f(C)$ and $C_{44} = ff'$.

Equation (21) leads to

$$f^2 = \frac{a}{(n^2-1)}C^{2-2n} + LC^{-4n} \tag{22}$$

where $a = \frac{\alpha^2}{l^2}$ and L is constant of integration.

Equation (22) leads to

$$\frac{dC}{\sqrt{(\frac{a}{n^2-1})C^{2-2n} + LC^{-4n}}} = dt \tag{23}$$

Hence the metric (1) leads to the form

$$ds^2 = -\left(\frac{dt}{dC}\right)^2 dC^2 + l^2 C^{2n} dx^2 + C^{2n} e^{-2\alpha x} dy^2 + C^2 dz^2 \tag{24}$$

The metric (24) after suitable transformation leads to the form

$$ds^2 = -\frac{dT^2}{(\frac{a}{n^2-1})T^{2-2n} + LT^{-4n}} + l^2 T^{2n} dx^2 + T^{2n} e^{-2\alpha x} dy^2 + T^2 dz^2 \tag{25}$$

where $C = T$.

4 Some Physical and Geometrical Properties

By (9), we have

$$\rho_4 + (\rho + p)\left(\frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C}\right) = 0 \tag{26}$$

The barotropic fluid condition leads to

$$p = \gamma\rho \tag{27}$$

Now using (14), (15) and (27) into (9) we have

$$\frac{\rho_4}{\rho} = -(1+\gamma)(2n+1)\frac{C_4}{C} \tag{28}$$

which leads to

$$\rho = \frac{b}{T^{(2n+1)(\gamma+1)}} \tag{29}$$

where $(2n + 1) > 0$ and b is a constant of integration.

Now using condition (27), (7) leads to

$$\frac{3}{4}\beta^2 = \frac{A_4}{A}\left(\frac{B_4}{B} + \frac{C_4}{C}\right) + \frac{B_4 C_4}{BC} - \frac{\alpha^2}{A^2} - K\rho \tag{30}$$

which leads to

$$\frac{3}{4}\beta^2 = (n^2 + 2n)\frac{C_4^2}{C^2} - \frac{\alpha^2}{l^2 C^{2n}} - \frac{Kb}{C^{(2n+1)(\gamma+1)}} \tag{31}$$

which again leads to

$$\frac{3}{4}\beta^2 = \frac{L(n^2+2n)}{T^{4n+2}} - \frac{(2n+1)a}{(1-n^2)T^{2n}} - K\frac{b}{T^{(2n+1)(\gamma+1)}} \tag{32}$$

The pressure is given by

$$p = \frac{\gamma b}{T^{(2n+1)(\gamma+1)}} \tag{33}$$

Also we have

$$\rho + p = \frac{(1+\gamma)b}{T^{(2n+1)(\gamma+1)}}, \tag{34}$$

$$\rho + 3p = \frac{(1+3\gamma)b}{T^{(2n+1)(\gamma+1)}} \tag{35}$$

Expansion θ is given by

$$\theta = \frac{A_4}{A} + \frac{B_4}{B} + \frac{C_4}{C} \tag{36}$$

which leads to

$$\theta = (2n+1)\frac{C_4}{C} \tag{37}$$

which again leads to

$$\theta = \frac{(2n+1)}{T^{2n+1}}\sqrt{L + \frac{a}{1-n^2}T^{2n+2}} \tag{38}$$

The shear tensor σ is given by

$$\sigma^2 = \frac{1}{3}\left[\frac{A_4^2}{A^2} + \frac{B_4^2}{B^2} + \frac{C_4^2}{C^2} - \frac{A_4 B_4}{AB} - \frac{B_4 C_4}{BC} - \frac{A_4 C_4}{AC}\right] \tag{39}$$

which leads to

$$\sigma = \frac{(n-1)}{\sqrt{3}T^{2n+1}}\sqrt{L + \frac{a}{1-n^2}T^{2n+2}} \tag{40}$$

where $n > 1$.

5 Special Case

To get the deterministic model in terms of cosmic time t, we put $L = 0$ so (23) leads to

$$\int C^{n-1}dC = \sqrt{\frac{a}{(n^2-1)}}\int dt$$

which leads to

$$C^n = (Nt + M) \tag{41}$$

where $N = n\sqrt{\frac{a}{n^2-1}}$ and M is a constant of integration. Also

$$B = (Nt + M), \tag{42}$$

$$A = l(Nt + M) \tag{43}$$

Using values of A, B and C, the metric (1) leads to

$$ds^2 = -dt^2 + l^2(Nt+M)^2 dx^2 + (Nt+M)^2 e^{-2\alpha x} dy^2 + (Nt+M)^{2/n} dz^2 \tag{44}$$

Using (29) energy density ρ, in this case is given by

$$\rho = \frac{b}{(Nt+M)^{(2n+1)(\gamma+1)/n}} \tag{45}$$

The isotropic pressure is given by

$$p = \gamma \frac{b}{(Nt+M)^{(2n+1)(\gamma+1)/n}} \tag{46}$$

Also

$$\rho + p = \frac{(\gamma+1)b}{(Nt+M)^{(2n+1)(\gamma+1)/n}} \tag{47}$$

and

$$\rho + 3p = \frac{(1+3\gamma)b}{(Nt+M)^{(2n+1)(\gamma+1)/n}} \tag{48}$$

The expansion θ and shear σ are given by

$$\theta = \frac{(2n+1)N}{n(Nt+M)}, \tag{49}$$

$$\sigma = \frac{(n-1)N}{\sqrt{3}n(Nt+M)} \tag{50}$$

Equations (49) and (50) lead to

$$\frac{\sigma}{\theta} = \frac{(n-1)}{\sqrt{3}(2n+1)} \tag{51}$$

which is a constant.

The displacement vector β is given by

$$\frac{3}{4}\beta^2 = (n^2+2n)\frac{C_4^2}{C^2} - \frac{\alpha^2}{l^2 C^{2n}} - \frac{Kb}{C^{(2n+1)(\gamma+1)}} \tag{52}$$

which leads to

$$\frac{3}{4}\beta^2 = \frac{(n^2+2n)N^2}{n^2(Nt+M)^2} - \frac{\alpha^2}{l^2(Nt+M)^2} - \frac{Kb}{(Nt+M)^{(2n+1)(\gamma+1)/n}} \tag{53}$$

which again leads to

$$\frac{3}{4}\beta^2 = \frac{(2n+1)N^2}{n^2(Nt+M)^2} - \frac{Kb}{(Nt+M)^{(2n+1)(\gamma+1)/n}} \tag{54}$$

The Scale factor R is given by

$$R^3 = ABCe^{-\alpha x}$$

which leads to

$$R^3 = le^{-\alpha x}(Nt+M)^{(2n+1)/n} \tag{55}$$

The deceleration parameter q is given by

$$q = -\frac{R_{44}/R}{R_4^2/R^2}$$

which leads to

$$q = -\frac{(n+1-2n^2)}{(2n+1)^2} \tag{56}$$

Hence

$$q < 0 \text{ when } \frac{2n^2}{n+1} < 1,$$

$$q > 0 \text{ when } \frac{2n^2}{n+1} > 1$$

6 Conclusion

The reality conditions $\rho + p > 0$, $\rho + 3p > 0$ given by Ellis [29] are satisfied when $b > 0$. For the model (25), $\frac{\sigma}{\theta}$ remains constant throughout. Hence anisotropy is maintained throughout. For $n = 1$, the model isotropizes. The model (25) starts with a big-bang at $T = 0$ and the expansion in the model decreases as time increases. $\rho \to \infty$ when $T \to 0$ and $(2n+1) > 0$ and $\rho \to 0$ when $T \to \infty$ and $(2n+1) > 0$. The spatial volume increases as time increases when $(2n+1) > 0$. The model (25) has Point Type singularity at $T = 0$ when $n > 0$ (MacCallum [30]). The gauge function $\beta = 0$ for $n = -1/2$ and $b = 0$.

The reality conditions $\rho + p > 0$, $\rho + 3p > 0$ given by Ellis [29] for the model (44) are satisfied when $b > 0$. The model (44) starts with a big-bang at $t = -M/N$ and the expansion in the model decreases as time increases. The expansion in the model stops at $n = -1/2$. The model (44) has Point Type singularity at $t = -M/N$ when $n > 0$ and it has Cigar Type singularity at $t = -M/N$ when $n < 0$ (MacCallum [30]). Since $\frac{\sigma}{\theta} \neq 0$, hence the anisotropy is maintained throughout. For $n = 1$, the model represents an isotropic universe. The deceleration parameter $q < 0$ when $\frac{2n^2}{n+1} < 1$ i.e. the model represents an expanding universe and $q > 0$ if $\frac{2n^2}{n+1} > 1$ i.e. in this case, the model represents a decelerating universe. The spatial volume (R^3) increases as time increases when $n > -1/2$. The gauge function β vanishes for $n = -1/2$ and $b = 0$.

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